THE COLLECTED MATHEMATICAL PAPERS OF ARTHUR CAYLEY.

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This republication by the Cambridge University Press of Cayley's papers in collected form is the most fitting monument of his splendid fame. He must ever rank as one of the greatest mathematicians of all time. Cayley exceedingly appreciated this action of the Syndics of the Press, and seven of the large quarto volumes appeared under his own editorship. As to what these 13 volumes contain, it seems vain to attempt even a summary. They cover the whole range of pure mathematics, algebra, analysis, mathematical astronomy, dynamics, and in particular groups, quadratic forms, quantics, etc., etc.

Though abreast of Sylvester as an analyst, he was, what Sylvester was not, also a geometer. Again and again we find the pure geometric methods of Poncelet and Chasles, though perhaps not full assimilation of that greater one than they, who has now absorbed them—Von Staudt. Cayley not only made additions to every important subject of pure mathematics, but whole new subjects, now of the most importance, owe their existence to him. It is said that he is actually now the author most frequently quoted in the living world of mathematicians.

His name is perhaps most closely linked with the word *invariant*, due to his great brother-in-arms, Sylvester. Boole in 1841 had shown the invariance of all discriminants and given a method of deducing some other such functions. This paper of Boole's suggested to Cayley the more general question, to find "all the derivations of any number of functions which have the property of preserving their form unaltered after any linear transformation of the variables." His first results, relating to what we now call invariants, he published in 1845. A second set of results, relating to what Sylvester called covariants, he published in 1846. Not until four or five years later did Sylvester take up this matter, but then came such a burst of genius that after his series of publications in 1851-54 the giant theory of Invariants and Covariants was in the world completely equipped.

The check came when Cayley, in his second Memoir on Quantics, came to the erroneous conclusion that the number of the asyzygetic invariants of binary quantics beyond the sixth order was infinite, "thereby," as Sylvester says, "arresting for many years the progress of the triumphal car which he had played a principal part in setting in motion." The passages supposed to prove this are marked "*incorrect*" in the Collected Mathematical Papers. But this error was not corrected until 1869, [Crelle, vol. 69., pp. 323-354] by Gordan in his Memoir [dated 8th June, 1868], 'Beweis dass jede Covariante und Invariante einer

binaeren Form eine ganze Function mit numerischen Coefficienten einer endlichen Anzahl solcher Formen ist.'

Cayley at once returned to the question, found the source of his mistake, the unspected and so neglected interdependence of certain syzygies, and devoted his Ninth Memoir on Quantics (7th April, 1870) to the correction of his error and a further development of the theory in the light of Gordan's results.

The whole of this primal theory of invariants may now be regarded as a natural and elegant application of Lie's theory of continuous groups. The differential parameters, which in the ordinary theory of binary forms enable us to calculate new invariants from known ones, appear in a simple way as differential invariants of certain linear groups. The Lie theory may be illustrated by a simple example :

Consider the binary quadratic form

$$f \equiv a_0 x^2 + 2a_1 xy + a_2 y^2 .$$

Applying to f the linear transformation

$$(1) \qquad x = \alpha x' + \beta y', \quad y = \gamma x' + \delta y',$$

we obtain the quadratic form

$$f' \equiv a'_0 x'^2 + 2a'_1 x'y' + a'_2 y'^2 ,$$

where the coefficients are readily found to be

$$(2) \qquad \begin{cases} a'_0 = \alpha^2 a_0 + 2\alpha\gamma a_1 + \gamma^2 a_2 , \\ a'_1 = \alpha\beta a_0 + (\alpha\delta + \beta\gamma) a_1 + \gamma\delta a_2 , \\ a'_2 = \beta^2 a_0 + 2\beta\delta a_1 + \delta^2 a_2 . \end{cases}$$

We may easily verify the following identity :

$$a'_0 a'_2 - a'^2_1 = (\alpha\delta - \beta\gamma)^2 (a_0 a_2 a_1^2).$$

Hence $a_0 a_2 - a_1^2$ is an invariant of the form f. In the group theory it is an invariant of the group of linear homogeneous transformations (2) on the three parameters a_0, a_1, a_2.

The only covariant of f is known to be f itself. In the Lie theory it appears as the *invariant* of a linear homogeneous group on five variables. x, y, a_0, a_1, a_2, the transformations being defined by the equations (2) together with (1) when inverted.

In general, the invariants of a binary form of degree n are defined by a linear homogeneous group on its $n+1$ coefficients, its covariants by a group on $n+3$ variables.

As in all problems in continuous groups, the detailed developments are greatly simplified by employing the infinitesimal transformations of the groups

concerned. It is readily proven by the group theory that all invariants and covariants are expressible in terms of a finite number of them. This result is, however, not equivalent to the algebraic result that all rational integral invariants (including covariants) are expressible rationally and integrally in terms of a finite number of such invariants.

Twenty years ago, in my "Bibliography of Hyper-Space and Non-Euclidean Geometry," (American Journal of Mathematics, Vol. I., Nos. 2 and 3, 1878) I cited seven of Cayley's papers written before 1873 :

I. Chapters in the Analytical Geometry of (n) Dimensions. Camb. Math. Jour., Vol, IV., 1845, pp. 119-127.

II. Sixth Memoir on Quantics. Phil. Trans., vol. 149. pp. 61-90, (1859).

III. Note on Lobatchewsky's Imaginary Geometry. Phil. Mag. XXIX,. pp. 231-233, (1865).

IV. On the rational transformation between two spaces. Lond. Math. Soc Proc. III., pp. 127-180, (1869-71).

V. A Memoir on Abstract Geometry. Phil. Trans. CLX., pp. 51-63, (1870).

VI. On the superlines of a quadric surface in five dimentional space. Quarterly Journ., Vol. XII., pp. 176-180, (1871-72).

VII. On the Non-Euclidean Geometry. Clebsch. Math. Ann. V., pp. 630-634, (1872).

Four of these pertain to Hyper-Space, and in that Bibliography I quoted Cayley as to its geometry as follows:

"The science presents itself in two ways,—as a legitimate extention of the ordinary *two-* and *three-*dimensional geometries; and as a need in these geometries and in analysis generally. In fact, whenever we are concerned with quantities connected together in any manner, and which are, or are considered as variable or determinable, then the nature of the relation between the quantities is frequently rendered more intelligible by regarding them (if only two or three in number) as coördinates of a point in a plane or in space : for more than three quantities there is, from the greater complexity of the case, the greater need of such a representation : but this can only be obtained by means of the notion of a space of the proper dimensionality : and to use such a representation, we require the geometry of such space. An important instance in plane geometry has actually presented itself in the question of the determination of the number of the curves which satisfy given conditions : the conditions imply relations between the coefficients in the equation of the curve ; and for the better understanding of these relations it was expedient to consider the coefficients as the coördinates of of a point in a space of the proper dimensionality."

For a dozen years after it was written, the Sixth Memoir on Quantics would not have been enumerated in a Bibliography of non-Euclidean geometry, for its author did not see that it gave a generalization which was identifiable with that initiated by Bolyai and Lobachévski, though afterwards, in his Address to the British Association, 1883, he attributes the fundamental idea involved to

Riemann, whose paper was written in 1854. Says Cayley: "In regarding the physical space of our experience as possibly non-Euclidean, Riemann's idea seems to be that of modifying the notion of distance, not that of treating it as a locus in four dimensional space."

What the Sixth Memoir was meant to do was to base a generalized theory of metrical geometry on a generalized definition of distance. As Cayley himself says: " . . . the theory in effect is, that the metrical properties of a figure are not the properties of the figure considered *per se* apart from everything else, but its properties when considered in connection with another figure, viz., the conic termed the absolute."

The fundamental idea that a metrical property could be looked at as a projective property of an extended system had occurred in the French school of geometers. Thus Laguerre (1853) so expresses an angle. Cayley generalized this French idea, expressing all metrical properties as projective relations to a fundamental configuration.

We may illustrate by tracing how Cayley arrives at his projective definition of distance. Two projective primal figures of the same kind of elements and both on the same bearer are called conjective. When in two conjective primal figures one particular element has the same mate to whichever figure it be regarded as belonging, then every element has this property. Two conjective figures such that the elements are mutually paired (coupled) form an involution. If two figures forming an involution have self correlated elements, these are called the double elements of the involution. An involution has at most two double elements; for were three self-correlated, all would be self-correlated. If an involution has two double elements these separate harmonically any two coupled elements. An involution is completely determined by two couples.

From all this it follows that two point-pairs A, B and A_1, B_1 define an involution whose double points D, D_1 are determined as that point-pair which is harmonically related to the two given point-pairs. Let the pair A, B be fixed and called the Absolute. Two new points A_1, B_1 are said (by definition) to be equidistant from a double point D defined as above. D is said to be the 'center' of the pair A_1, B_1. Inversely, if A_1 and D be given, B_1 is uniquely determined. Thus starting from two points P and P_1, we determine P_2 such that P_1 is the center of P and P_2, then determine P_3 so that P_2 is the center of P_1 and P_3, etc.; also in the opposite direction, we determine an analogous series of points P_{-1}, P_{-2}, ... We have therefore a series of points

$$\ldots\ldots, P_{-2}, P_{-1}, P, P_1, P_2, P_3, \ldots\ldots$$

at 'equal intervals of distance.' Taking the points P, P_1 to be indefinitely near to each other, the entire line will be divided into a series of equal infinitesimal elements.

In determining an analytic expression for the distance of the two points, Cayley introduced the inverse cosine of a certain function of the coördinates, but

in the Note which he added in the Collected Papers he recognizes the improvement gained by adopting Klein's assumed definition for the distance of any two points P, Q: dist. $(PQ)=c \log\frac{AP.BQ}{AQ.BP}$, where A, B are the two fixed points giving the Absolute.

This definition preserves the fundamental relation

$$\text{dist. }(PQ)+\text{dist. }(QR)=\text{dist. }(PR).$$

In this note (Col. Math. Papers, Vol. 2, p. 604) Cayley discusses the question whether the new definitions of distance depend upon that of distance in the ordinary sense, since it is obviously unsatisfactory to use one conception of distance in defining a more general conception of distance. His earlier view was to regard coördinates "not as distances or ratios of distances, but as an assumed fundamental notion, not requiring or admitting of explanation." Later he regarded them as "mere numerical values, attached arbitrarily to the point, in such wise that for any given point the ratio $x:y$ has a determinate numerical value," and inversely.

But in 1871 Klein had explicitly recognized this difficulty and indicated its solution. He says: "The cross ratios (the sole fixed elements of projective geometry) naturally must not here be defined, as ordinarily happens, as ratios of sects, since this would assume the knowledge of a measurement. In von Staudt's Beitraegen Zur Geometrie der Lage (§ 27, n. 393), however, the necessary materials are given for defining a cross ratio as a pure number. Then from cross ratios we may pass to homogeneous point—and plane—coördinates, which indeed are nothing else than the relative values of certain cross ratios, as von Staudt has likewise shown (Beitrage, § 29, n. 411)."

This solution was not satisfactory to Cayley, who did not think the difficulty removed by the observations of von Staudt that the cross ratio (A, B, P, Q) has "independently of any notion of distance the fundamental properties of a numerical magnitude, viz., any two such ratios have a sum and also a product, such sum and product being each of them a like ratio of four points determinable by purely descriptive construction."

Consider, for example, the product of the ratios (A, B, P, Q) and (A', B', P', Q'). We can construct a point R such that $(A', B', P', Q')=(A, B, Q, R)$. The product of (A, B, P, Q) and (A, B, Q, R) is said to be (A, B, P, R). This last definition of a product of two cross ratios, Cayley remarks, is in effect equivalent to the assumption of the relation

$$\text{dist. }(PQ)+\text{dist. }(QR)=\text{dist. }(PR).$$

The original importance of this memoir to Cayley lay entirely in its exhibiting metric as a branch of descriptive geometry. That this generalization of distance gave pangeometry was first pointed out by Klein in 1871. Klein showed that if Cayley's Absolute be real, we get Bolyai's system; if it be imagi-

nary, we get either spheric or a new system called by Klein single elliptic; if the Absolute be an imrginary point pair, we get parabolic geometry, and if, in particular, the point pair be the circular points, we get ordinary Euclid.

It is maintained by B. A. W. Russell in his powerful Essay on the Foundations of Geometry (Cambridge, 1897) "that the reduction of metrical to projective properties, even when, as in hyperbolic geometry, the Absolute is real, is only apparent, and has merely a technical validity."

Cayley first gave evidence of acquaintance with non-Euclidean geometry in 1865 in the paper in the Philosophical Magazine above mentioned. Though this is six years after the Sixth Memoir, and though another six was to elapse before the two were connected, yet this is, I think, the very first appreciation of Lobachévski in any mathematical journal. Baltzer has received deserved honor for in 1866 calling the attention of Hoüel to Lobachévski's 'Geometrische Untersuchungen,' and from the spring thus opened actually flowed the flood of ever broadening non-Euclidean research. But whether or not Cayley's path to these gold-fields was ever followed by anyone else, still he had therein marked out a claim for himself a whole year before the others.

In 1872 after the connection with the Sixth Memoir had been set up, Cayley takes up the matter in his paper in the Mathematische Annalen 'On the Non-Euclidean Geometry,' which begins as follows: "The theory of the Non-Euclidian Geometry as developed in Dr. Klein's paper, 'Ueber die Nicht-Euclidische Geometrie,' may be illustrated by showing how in such a system we actually measure a distance and an angle and by establishing the trigonometry of such a system. I confine myself to the 'hyperbolic' case of plane geometry; viz., the absolute is here a real conic, which for simplicity I take to be a circle; and I attend to the points *within* the circle. I use the simple letters, a, A, . . to denote (linear or angular) distances measured in the ordinary manner; and the same letters with a superscript stroke $\bar{a}$, $\bar{A}$, . . to denote the same distances measured according to the theory. The radius of the absolute is for convenience taken to $=1$, the distance of any point from the center can therefore be represented as the sine of an angle.

The distance $\overline{BC}$, or say $\bar{a}$, of any two points B, C is by definition as follows:

$$\bar{a}=\tfrac{1}{2}\log\frac{BI.CJ}{BJ.CI}$$

(where I, J are the intersections of the line BC with the circle)."

As for the trigonometry, "the formulae are in fact similar to those of spherical trigonometry with only $\cosh\bar{a}$, $\sinh\bar{a}$, etc., instead of $\cos a$, $\sin a$, etc."

Cayley returned again to this matter in his celebrated Presidential Address to the British Association (1883), saying there: "It is well known that Euclid's twelfth axiom, even in Playfair's form of it, has been considered as needing demonstration; and that Lobatschewsky constructed a perfectly consistent theory wherein this axiom was assumed not to hold good, or say a system of non-Euclidean plane geometry. There is a like system of non-Euclidean solid

geometry." "But suppose the physical space of our experience to be thus only approximately Euclidean space, what is the consequence which follows?"

The very next year this ever interesting subject recurs in the paper (May 27, 1884) "On the Non-Euclidean Plane Geometry." "Thus the geometry of the pseudo-sphere, using the expression straight line to denote a geodesic of the surface, is the Lobatschewskian geometry; or rather I would say this in regard to the metrical geometry, or trigonometry, of the surface; for in regard to the descriptive geometry, the statement requires some qualification . . . this is not identical with the Lobatschewskian geometry, but corresponds to it in a manner such as that in which the geometry of the surface of the circular cylinder corresponds to that of the plane.

I would remark that this realization of the Lobatschewskian geometry sustains the opinion that Euclid's twelfth axiom is undemonstrable."

But here this necessarily brief notice must abruptly stop.

Cayley in addition to his wondrous originality was assuredly the most learned and erudite of mathematicians. Of him in his science it might be said, he knew everything, and he was the very last man who ever will know everything. His was a very gentle, sweet character. Sylvester told me he never saw him angry but once, and that was (both were practicing law!) when a messenger broke in on one of their interviews with a mass of legal documents, new business for Cayley. In an excess of disgust, Cayley dashed the documents upon the floor.

Austin, Texas, February, 1899.

NOTE ON SPHERICAL GEOMETRY.

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DEFINITION. Two arcs of great circles drawn from the vertex of a spherical triangle making equal angles with the spherical bisector of the angle at that vertex are called *isogonal conjugate arcs.* If three arcs drawn through the vertices of a spherical triangle are concurrent, their *isogonal conjugates* with respect to the angles at these vertices are also concurrent.

Let the arcs AM_a, BM_b, CM_c be concurrent at M. To prove that their isogonal conjugates AK_a, BK_b, CK_c are concurrent.

Fig. 1. Let $BM_a=a_1$, $CM_a=a_2$, $CM_b=b_1$, $AM_b=b_2$, $AM_c=c_1$, $BM_c=c_2$, $BK_a=a_3$, $CK_a=a_4$, $CK_b=b_3$, $AK_b=b_4$, $AK_c=c_3$, $BK_c=c_4$.

$\angle CMM_b=x$, $\angle CMM_a=y$, $\angle BMM_a=z$, $\angle CAM_a=\angle BAK_a=\theta$, $\angle BAM_a=$

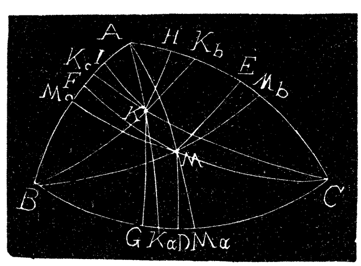

Fig. 1.